Non-Traditional Composite/Operation Limits

The commonly memorized limit of composition rule generally is presented as something like this:

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) \text{ provided that } \lim_{x\to a} g(x) = L \text{ and } f(x) \text{ is continuous at } x = L.$$

This is useful, but like many $if - then$ statements it gets misread as an $if\ and\ only\ if$ statement. If the $f(x)$ is not continuous at $x = L$ or if the limit of $g(x)$ does not exist at $x = a$, then we cannot apply the property. It does not tell us anything conclusive about the existence of $\lim_{x\to a} f(g(x))$.

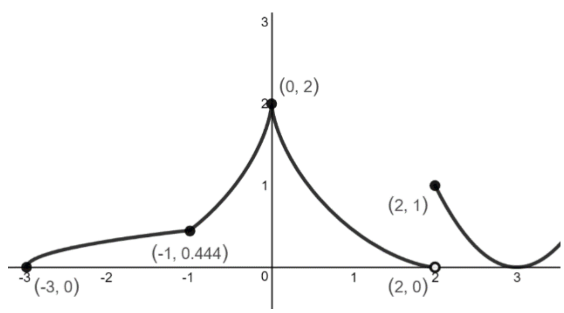
Similar discussions follow from all the 'standard' limit properties. E.g., If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$, then $\lim_{x\to a}(f(x) + g(x)) = L_1 + L_2$. However, if $\lim_{x\to a} f(x)$ and/or $\lim_{x\to a} g(x)$ don't exist, we cannot draw any conclusion about $\lim_{x\to a}(f(x) + g(x)) = L_1 + L_2$.

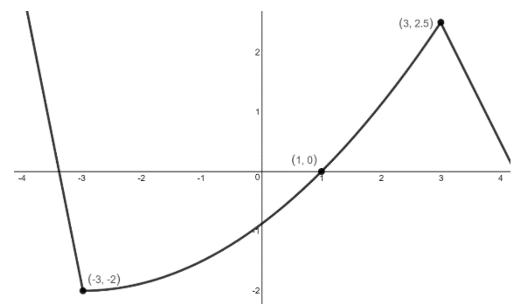
What can we do when the traditional limit properties don't apply? Well they also can be applied to one-sided limits. The most direct and effective route to exploring the limit is to remember another property of limits that IS an $if\ and\ only\ if$ statement:

$$\text{Iff } \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L, \text{ then } \lim_{x\to a} f(x) = L.$$

By exploring the right-hand and left-hand compositions or operations, we can often find if the limit being explored exists or not.

Consider: the graph of $f(x)$ the graph of $g(x)$



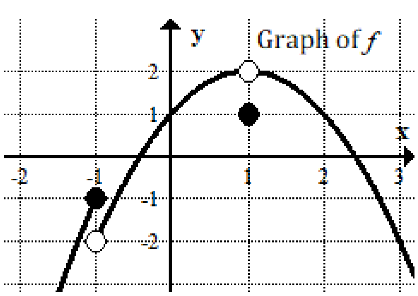


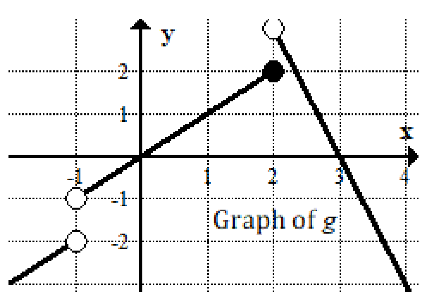
Find (a) $\lim_{x\to 1} f(g(x))$ (b) $\lim_{x\to 2}[f(x) \cdot g(x)]$ (c) $\lim_{x\to 0} f(f(x))$

(a) For this composition, traditional rules apply.

(b) While the limit of $f(x)$ doesn't exist at $x = 2$, explore the left-hand and right-hand limits of the products.

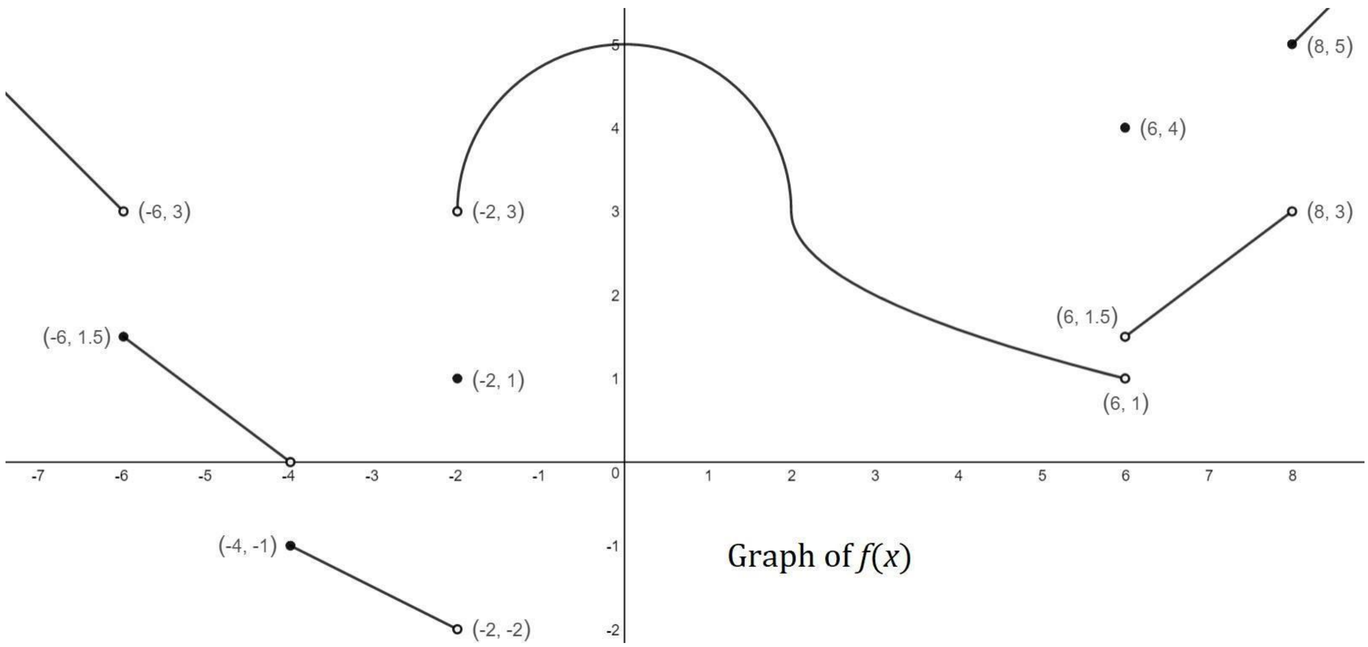
(c) It is important to consider from what direction the inner limit value is approaching 2. [Hint: Sometimes a tabular approach helps see what's happening, even if you don't know the actual function.]

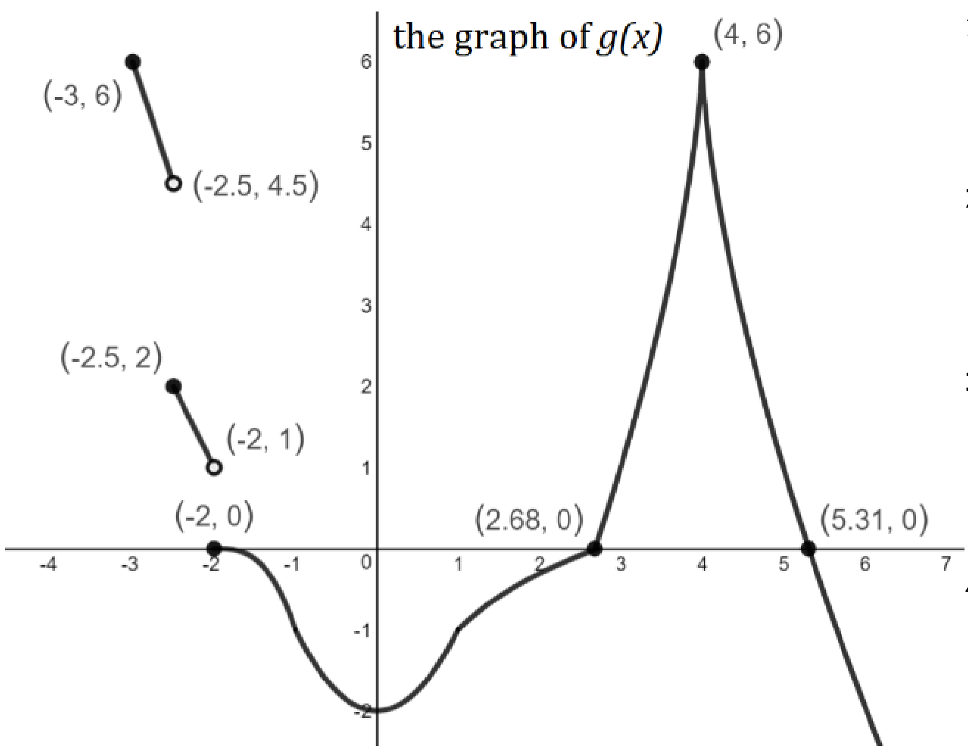




Find $\lim_{x\to -1}[f(x) + g(x)]$.

Find $\lim_{x\to 1}[f(x) + g(x+1)]$.





1. Find $f(g(4))$.

2. Find $\lim_{x\to 4} f(g(x))$.

3. Find $f(g(0))$.

4. Find $\lim_{x\to 0} f(g(x))$.

5. Find $\lim_{x\to 6^-} g(1-f(x))$.

6. Find $\lim_{x\to 3^+} f(15-x^2)$.

7. Find $\lim_{x\to 2^-} g(2-x^2)$.

8. Find $\lim_{x\to -2^-} f(f(x))$.

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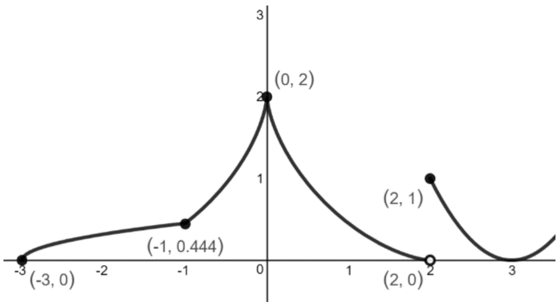
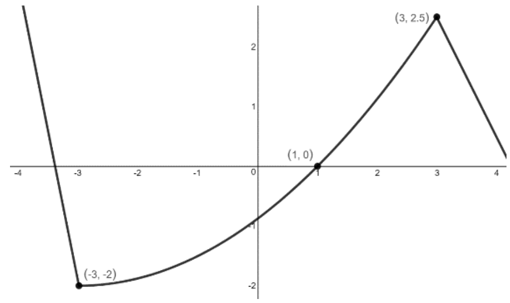
Consider: the graph of $f(x)$ the graph of $g(x)$





Find (a) $\lim_{x\to 1} f(g(x))$ (b) $\lim_{x\to 2}[f(x)\cdot g(x)]$ (c) $\lim_{x\to 0} f(f(x))$

(a) For this composition, traditional rules apply.

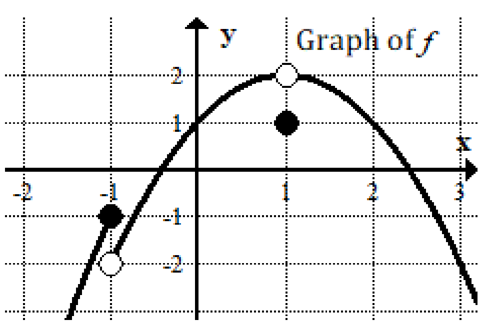
$$\lim_{x\to 1} f(g(x)) = f(\lim_{x\to 1} g(x)) = f(0) = 2$$

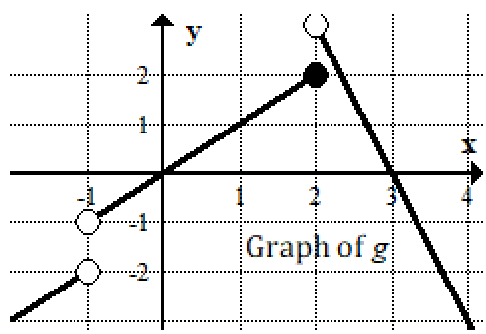
(b) While the limit of $f(x)$ doesn't exist at $x = 2$, explore the left-hand and right-hand limits of the products.

$\lim_{x\to 2^-}[f(x)\cdot g(x)] = 0\cdot 1 = 0$ $\lim_{x\to 2^+}[f(x)\cdot g(x)] = 1\cdot 1 = 1$ $\lim_{x\to 2}[f(x)\cdot g(x)]$ does not exist.

(c) It is important to consider from what direction the inner limit value is approaching 2. [Hint: Sometimes a tabular approach helps see what's happening, even if you don't know the actual function.]

$u = \lim_{x\to 0} f(x) = 2^-$ (from lower values) $\lim_{x\to 0} f(f(x)) = \lim_{u\to 2^-} f(u) = 0$





Find $\lim_{x\to -1}[f(x) + g(x)]$.

$\lim_{x\to -1^-}[f(x) + g(x)] = [-1 + (-2)] = -3$

$\lim_{x\to -1^+}[f(x) + g(x)] = [-2 + (-1)] = -3$

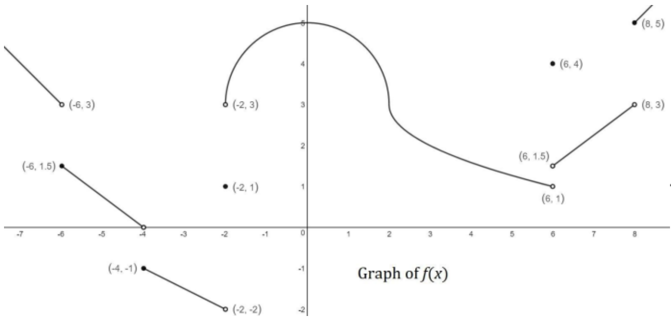
$\lim_{x\to -1}[f(x) + g(x)] = -3$

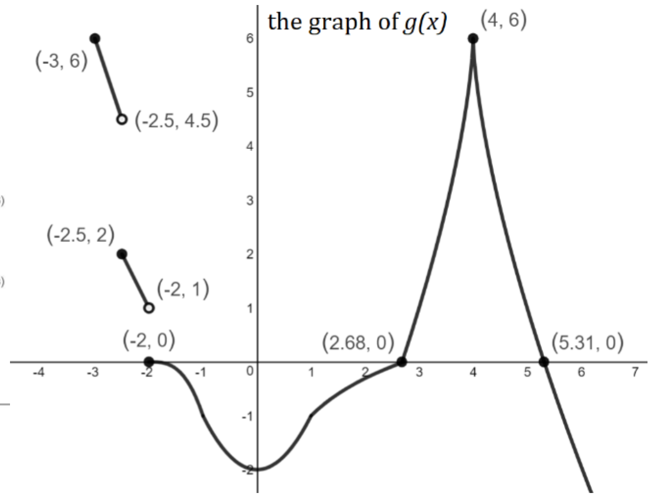
Find $\lim_{x\to 1}[f(x) + g(x+1)]$.

$\lim_{x\to 1^-}[f(x) + g(x+1)] = [2 + 2] = 4$

$\lim_{x\to 1^+}[f(x) + g(x+1)] = [2 + 3] = 5$

$\lim_{x\to 1}[f(x) + g(x+1)]$ does not exist





1. Find $f(g(4))$.

$f(g(4)) = f(6) = 4$

2. Find $\lim_{x\to 4} f(g(x))$.

$u = \lim_{x\to 4} g(x) = 6^-$ from below

$\lim_{u\to 6^-} f(u) = 1$

3. Find $f(g(0))$.

$f(-2) = 1$

4. Find $\lim_{x\to 0} f(g(x))$.

$u = \lim_{x\to 0} g(x) = -2^+$ from above

$\lim_{u\to -2^+} f(u) = 3$

5. Find $\lim_{x\to 6^-} g(1 - f(x))$.

$u = \lim_{x\to 6^-} f(x) = 1^+$

$v = \lim_{u\to 1^+} (1 - 1^+) = 0^-$

$\lim_{v\to 0^-} g(v) = -2$ (either side)

6. Find $\lim_{x\to 3^+} f(15 - x^2)$.

$u = \lim_{x\to 3^+} x^2 = 9^+$

$v = \lim_{u\to 9^+} (15 - u) = 6^-$

$\lim_{v\to 6^-} f(v) = 1$

7. Find $\lim_{x\to 2^-} g(2 - x^2)$.

$u = \lim_{x\to 2^-} x^2 = 4^-$

$v = \lim_{u\to 4^-} (2 - u) = -2^+$

$\lim_{v\to -2^+} g(v) = 0$

8. Find $\lim_{x\to -2^-} f(f(x))$.

$u = \lim_{x\to -2^-} f(x) = -2^+$

$\lim_{u\to -2^+} f(u) = 3$